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$$P_C^2 = \begin{pmatrix} p^8 + 4p^7q + 2p^6q^2 & 4p^7q + 4p^6q^2 & 2p^6q^2 \\ p^8 + p^7 & 4p^7 + p^6 & 2p^6 \\ p^8 & 4p^7 + p^6 & 2p^6 \end{pmatrix}$$

$$R_C(2) = p^8 + 8p^7q + 8p^6q^2$$

$$P_C^3 = \begin{pmatrix} p^{12} + 8p^{11}q + 8p^{10}q^2 & 4p^{11}q + 20p^{10}q^2 + 12p^9 & 2p^{10}q^2 + 8p^9q^3 + 4p^8q^4 \\ p^{12} + 5p^{11}q + 3p^{10}q^2 & 4p^{11}q + 8p^{10}q^2 + p^9q^3 & 2p^{10}q^2 + 2p^9q^3 \\ p^{12} + 4p^{11}q + 2p^{10}q^2 & 4p^{11}q + 4p^{10}q^2 & 2p^{10}q^2 \end{pmatrix} \Rightarrow$$

$$R_C(3) = p^{12} + 12p^{11}q + 30p^{10}q^2 + 20p^9q^3 + 4p^8q^4$$

Conclusion:

In this paper, we compute the reliability of the 2-within-consecutive (2,2)-out-of-(m,n): F linear (rectangle) and circular (cylindrical) system using a Markov Chain. Furthermore, the computation process of the reliability of the cylindrical system is simpler than the rectangle system since the number of states of the Markov chain in the circular case is less than that of the linear case. We wish to generalize the used technique to find the reliability of the k-within consecutive **(r,s)** -out-of- **(m,n)** system.

$$P_L^2 = \begin{pmatrix} p^8+4p^7q+3p^6q^2 & 2p^7q+4p^6q^2 & 2p^7q+2p^6q^2 & 2p^6q^2 & p^6q^2 \\ p^8+2p^7q & 2p^7q+2p^6q^2 & 2p^7q+p^6q^2 & 2p^6q^2 & p^6q^2 \\ p^8+p^7q & 2p^7q+p^6q^2 & 2p^7q+p^6q^2 & 2p^6q^2 & p^6q^2 \\ p^8 & 2p^7q & 2p^7q & 2p^6q^2 & p^6q^2 \\ p^8 & 2p^7q & 2p^7q & 2p^6q^2 & p^6q^2 \end{pmatrix}$$

$$R_L(2) = p^8+8p^7q+12p^6q^2$$

$$P_L^3 = \begin{pmatrix} p^{12}+8p^{11}q+12p^{10}q^2 & 2p^{11}q+12p^{10}q^2+12p^9q^3 & 2p^{11}q+10p^{10}q^2+10p^9q^3 & 2p^{10}q^2+8p^9q^2+6p^8q^2 & p^{10}q^2+4p^9q^2+3p^8q^2 \\ p^{12}+6p^{11}q+6p^{10}q^2 & 2p^{11}q+8p^{10}q^2+3p^9q^3 & 2p^{11}q+6p^{10}q^2+2p^9q^3 & 2p^{10}q^2+4p^9q^2 & p^{10}q^2+2p^9q^2 \\ p^{12}+5p^{11}q+5p^{10}q^2 & 2p^{11}q+6p^{10}q^2+2p^9q^3 & 2p^{11}q+4p^{10}q^2+p^9q^3 & 2p^{10}q^2+2p^9q^2 & p^{10}q^2+p^9q^2 \\ p^{12}+4p^{11}q+3p^{10}q^2 & 2p^{11}q+4p^{10}q^2 & 2p^{11}q+2p^{10}q^2 & 2p^{10}q^2 & p^{10}q^2 \\ p^{12}+4p^{11}q+3p^{10}q^2 & 2p^{11}q+4p^{10}q^2 & 2p^{11}q+2p^{10}q^2 & 2p^{10}q^2 & p^{10}q^2 \end{pmatrix}$$

$$R_L(3) = p^{12}+12p^{11}q+37p^{10}q^2+34p^9q^2+9p^8q^3$$

Example 5.2: The reliability of 2 within (2,2) out of (3,4): F cylindrical system

$$\Theta_C^2 = \{\varnothing,1,2,3,4,13,24\}, \qquad [\varnothing]_C^2 \qquad [1]_C^2 = \{1,2,3,4\} \qquad [13]_C^2 = \{13,24\}$$

The transitive probability matrix P_C is

$$P_C = \left.\begin{array}{c} \begin{array}{cccc} [\varnothing]_C^2 & [1]_C^2 & [13]_C^2 & \Psi_C^2 \end{array} \\ \left(\begin{array}{cccc|c} [\varnothing]_C^2 & p^4 & 4p^3q & 2p^2q^2 & C_\varnothing \\ [1]_C^2 & p^4 & p^3q & 0 & C_1 \\ [13]_C^2 & p^4 & 0 & 0 & C_{13} \\ \hline \Psi_C^2 & 0 & 0 & 0 & 1 \end{array}\right) \end{array}\right\} \Rightarrow P_C' = \begin{pmatrix} p^4 & 4p^3q & 2p^2q^2 \\ p^4 & p^3q & 0 \\ p^4 & 0 & 0 \end{pmatrix}$$

for example:

$$A_C^{i+1}\left(1,[1]^2\right) = \left\{Z \in [1]_C^2 : f_C^\alpha(x) \notin \{1\}, \forall \alpha = 1,3,4\right\} = \{3\} \Rightarrow d_1^1 = 1 \Rightarrow P_C[1,1] = p^3q$$

$$A_C^{i+1}\left(13,[1]^2\right) = \left\{Z \in [1]_C^2 : f_C^\alpha(x) \notin \{13\}, \forall \alpha = 1,3,4\right\} = \varnothing \Rightarrow d_1^{13} \Rightarrow P_C[13,1] = 0$$

2 x 2 -out-of- **m x n** : F system in a Markov chain, ***it consists of the following steps:***

1. Consider the consecutive 2-out-of-n: F system.
2. Set the equivalence classes of $\Theta^2_{L(C)}$, which are the states of the Markov chain.
3. Calculate the transient probabilities matrix $P_{L(C)}$.
4. Remove the last row and last column of $P_{L(C)}$, to take only the functioning states, and rename it P'_L, and then compute $P'^{m}_{L(C)}$
5. The reliability of the 2-within (2,2)-out-of-(m,n): F linear (rectangle) and circular (cylindrical) is the sum the first row in $P'^{m}_{L(C)}$.

Illustrative Examples:

Example 5.1: The reliability of 2 within (2,2) out of (3,4): F rectangle system

$$\Theta^2_L = \{\varnothing, 1, 2, 3, 4, 13, 14, 24\}$$

$$[\varnothing]^2_L \qquad [1]^2_L = \{1,4\} \qquad [2]^2_L = \{2,3\} \qquad [13]^2_L = \{13,24\} \qquad [14]^2_L = \{14\}$$

The transitive probability matrix P_L is

$$P_L = \begin{array}{c} \text{classes} \\ [\varnothing]^2_L \\ [1]^2_L \\ [2]^2_L \\ [13]^2_L \\ [14]^2_L \\ \Psi^2_{L(C)} \end{array} \begin{array}{c} \begin{array}{cccccc} [\varnothing]^2_L & [1]^2_L & [2]^2_L & [13]^2_L & [14]^2_L & \Psi^2_{L(C)} \end{array} \\ \left(\begin{array}{ccccc|c} p^4 & 2p^3q & 2p^3q & 2p^2q^2 & p^2q^2 & C_\varnothing \\ p^4 & p^3q & 0 & 0 & 0 & C_1 \\ p^4 & p^3q & 0 & 0 & 0 & C_2 \\ p^4 & 0 & 0 & 0 & 0 & C_{13} \\ p^4 & 0 & 0 & 0 & 0 & C_{14} \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array} \Rightarrow P'_L = \begin{pmatrix} p^4 & 2p^3q & 2p^3q & 2p^2q^2 & p^2q^2 \\ p^4 & p^3q & 0 & 0 & 0 \\ p^4 & p^3q & 0 & 0 & 0 \\ p^4 & 0 & 0 & 0 & 0 \\ p^4 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For example

$$A^{i+1}_L\left(1,[1]^2\right) = \left\{Z \in [1]^2_L : |z - x| \geq 2\right\} = \{3\} \Rightarrow d^1_1 = 1 \Rightarrow P_L[1,1] = p^3q$$

$$A^{i+1}_L\left(1,[13]^2\right) = \left\{Z \in [13]^2_L : |z - x| \geq 2\right\} = \varnothing \Rightarrow d^{13}_1 = 0 \Rightarrow P_L[1,13] = 0$$

$$\forall W \in [X_u]^2_{L(C)} \Rightarrow p^i_W q^i_W = p_W q_W = p_{X_u} q_{X_u} = R(X_u), \text{and } \forall Z \in [X_v]^2_{L(C)} \Rightarrow$$

$$p^{i+1}_Z q^{i+1}_Z = p_Z q_Z = p_{X_v} q_{X_v} = R(X_v)$$

$$P_{L(C)}[X_u, X_v] = \frac{\sum_{W \in [X_u]^2_{L(C)}} \left(p^{n-d_{X_u}} q^{d_{X_u}} \sum_{Z \in A^{i+1}_{L(C)}\left(W,[X_v]^2\right)} p^{n-d_{X_v}} q^{d_{X_v}} \right)}{\sum_{W \in [X_u]^2_{L(C)}} p^{n-d_{X_u}} q^{d_{X_u}}}$$

$$P_{L(C)}[X_u, X_v] = \frac{\sum_{W \in [X_u]^2_{L(C)}} \left(R(X_u) \sum_{Z \in A^{i+1}_{L(C)}\left(W,[X_v]^2\right)} R(X_v) \right)}{\sum_{W \in [X_u]^2_{L(C)}} R(X_u)} = \frac{\left(\sum_{W \in [X_u]^2_{L(C)}} R(X_u) \right)\left(\sum_{Z \in A^{i+1}_{L(C)}\left(W,[X_v]^2\right)} R(X_v) \right)}{\sum_{W \in [X_u]^2_{L(C)}} R(X_u)}$$

$$= \sum_{Z \in A^{i+1}_{L(C)}\left(W,[X_v]^2\right)} R(X_v)$$

Since $W \in [X_u]^2_{L(C)} \Rightarrow \exists \alpha \in \mathbf{Z} \ni f^{\alpha}_{L(C)}(X_u) = W$, using Lemma 3.1, then

$$P_{L(C)}[X_u, X_v] = \sum_{Z \in A^{i+1}_{L(C)}\left(W,[X_v]^2\right)} R(X_v) = \sum_{f^{\alpha}(Z) \in f^{\alpha}_{L(C)}\left(A^{i+1}_{L(C)}\left(W,[X_v]^2\right)\right) = A^{i+1}_{L(C)}\left(X_u,[X_v]^2\right)} R(X_v) = d^{X_v}_{X_u} R(X_v)$$

Evaluation the proposed algorithm:

Few researchers used the Markov Chain to evaluate the exact reliability of the 2-dimensional linear (rectangular) & circular (cylindrical) k-within-consecutive- **r x s** -out-of- **m x n** : F system, Chang and Huang [4] used 2-dimensional discrete scan statistic to find the sample space of the columns of the rectangle system to define the states of a Markov chain, while the cylindrical system treated as an extension of the rectangle system with special assumption. Our proposed algorithm uses the equivalence relations for both the linear and circular consecutive 2-out-of-n: F system as in section 2 to reduce the state of the rows of the system to imbed the 2-within consecutive

$$P_{L(C)}\left[X_{s+1}, X_v\right] = \begin{cases} 0 & v \neq s+1 \\ 1 & v = s+1 \end{cases}.$$

2. For any i=1,2,..,m-1, u=0, i.e. $[X_0]^2_{L(C)} = [\varnothing]^2_{L(C)}$ then;

$$\sum_{v=0}^{s} P_{L(C)}\left[X_0, X_v\right] = \sum_{v=0}^{s} \frac{P\left(S_{i+1} = \left[X_v\right]^2_{L(C)}, S_i = \left[X_0\right]^2_{L(C)}\right)}{P\left(S_i = \left[X_0\right]^2_{L(C)}\right)} =$$

$$= \sum_{v=0}^{s} \frac{\left(p^i_{X_0}\right) P\left\{A_{L(C)}\left(X_0, \left[X_v\right]^2_{L(C)}\right)\right\}}{\left(p^i_{X_0}\right)} = \sum_{v=0}^{s} P\left\{A^{i+1}_{L(C)}\left(X_0, \left[X_v\right]^2_{L(C)}\right)\right\} =$$

$$= \sum_{v=0}^{s} P\left(S_{i+1} = \left[X_v\right]^2_{L(C)}\right) = P\left\{\Theta^2_{L(C)}\right\} = R^2_{L(C)} = R_{L(C)}(1)$$

According to theorem 3.1 in Koutras [6], if $\boldsymbol{\eth}_0{}^T = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$ is the initial probability, and $\mathbf{u} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 \end{bmatrix}^T$ is the row vector, then the reliability of the 2-within consecutive (2,2)-out-of-(m,n): F rectangle (cylindrical) system is

$$R_{L(C)}(m) = \boldsymbol{\eth}_0{}^T \left(\prod_{l=1}^{m} \mathbf{P} \right) \mathbf{u}$$

$$= \sum_{v=0}^{s} P^m_{L(C)}\left[X_0, X_v\right] = \sum_{\left[X_v\right]^2_{L(C)} \in \Theta^2_{L(C)}} P^m_{L(C)}\left[X_0, X_v\right] = R_{L(C)}(m)$$

3. If the components are i.i.d. $p^i_j = p : j = 1,2,...,n; i = 1,2,...,m$

$$P_{L(C)}\left[X_u, X_v\right] = \frac{\sum_{W \in \left[X_u\right]^2_{L(C)}} \left(p^i_W q^i_W \sum_{Z \in A^{i+1}_{L(C)}\left(W, \left[X_v\right]^2\right)} p^{i+1}_Z q^{i+1}_Z \right)}{\sum_{W \in \left[X_u\right]^2_{L(C)}} p^i_W q^i_W}$$

Using lemma 2.2

where $C_{X_u} = 1 - \sum_{v=0}^{s} d_{X_u}^{X_v} p^{n-d_{X_v}} q^{d_{X_v}}$ and $d_{X_u}^{X_v} = d_{A_{L(C)}^{i+1}\left(X_u,[X_v]^2\right)}$,
$\left(\forall X_u \in \Theta_{L(C)}^2 \Rightarrow d_{X_u}^{\varnothing} = d_{A_{L(C)}^{i+1}\left(X_u,[\varnothing]^2\right)} = d_{\varnothing} = 1\right)$

Proof:

1. Consider the system in the functioning state, and assume the state of the subsystem with i layer (circle) is represented by any element of the class $[X_u]_{L(C)}^2 \in \Theta_{L(C)}^2 : u = 0,1,\ldots,s$, and the state of the (i+1)th layer (circle) is any element of the class $[X_v]_{L(C)}^2 \in \Theta_{L(C)}^2 : v = 0,1,\ldots,s$, the probability $P_{L(C)}[X_u, X_v]$ where $i = 1,2,\ldots,m-1$, is

$$P_{L(C)}[X_u,X_v] = \frac{P\left(S_{i+1} = [X_v]_{L(C)}^2, S_i = [X_u]_{L(C)}^2\right)}{P\left(S_i = [X_u]_{L(C)}^2\right)} = \frac{\sum_{W \in [X_u]_{L(C)}^2} \left(p_W^i q_W^i P\left\{A_{L(C)}^{i+1}\left(W,[X_v]^2\right)\right\}\right)}{\sum_{W \in [X_u]_{L(C)}^2} p_W^i q_W^i}$$

$$= \frac{\sum_{W \in [X_u]_{L(C)}^2} \left(p_W^i q_W^i \sum_{Z \in A_{L(C)}^{i+1}\left(W,[X_v]^2\right)} p_Z^{i+1} q_Z^{i+1}\right)}{\sum_{W \in [X_u]_{L(C)}^2} p_W^i q_W^i}$$

For $u = 0,1,2,\ldots s, v = s+1$, the probability $P_{L(C)}[X_u, X_{s+1}]$ indicates that the system moves from the functioning state $[X_u]_{L(C)}^2 \in \Theta_{L(C)}^2$ to the failure state $[X_{s+1}]_{L(C)}^2 \in \Psi_{L(C)}^2$, according Markov chain properties $P_{L(C)}[X_u, X_{s+1}] = 1 - \sum_{v=0}^{s} P_{L(C)}[X_u, X_v]$.

For $u = s+1$, i.e. $[X_{s+1}]_{L(C)}^2 = \Psi_{L(C)}^2$ describes the system's breakdown, this level corresponds to an absorbing state , hence the probability representing the state of the ith layer (circle) as in definition 3.1, then

1. $P_{L(C)}\left[X_u, X_v\right]$ the probability that the subsystem moves from the state $\left[X_u\right]^2_{L(C)}$ with i layers (circles) to the state $\left[X_v\right]^2_{L(C)}$ with (i+1)th layers (circles) is

$$P_{L(C)}\left[X_u, X_v\right] = \begin{cases} \dfrac{\displaystyle\sum_{W \in [X_u]^2_{L(C)}} \left(p_W^i q_W^i \sum_{Z \in A^{i+1}_{L(C)}\left(Z, [X_v]^2\right)} p_Z^{i+1} q_Z^{i+1} \right)}{\displaystyle\sum_{W \in [X_u]^2_{L(C)}} p_W^i q_W^i} & u, v = 0, 1, 2, \ldots s \\ 1 - \displaystyle\sum_{v=0}^{s} P_{L(C)}\left[X_u, X_v\right] & u = 0, 1, 2, \ldots s, v = s+1 \\ 0 & u = s+1, v = 0, 1, \ldots, s \\ 1 & u = s+1, v = s+1 \end{cases}$$

2. The reliability of the 2-within (2,2)-out-of-(m,n): F linear (rectangle) and circular (cylindrical) system is

$$R_{L(C)}(m) = \sum_{[X_v]^2_{L(C)} \in \Theta^2_{L(C)}} P^m_{L(C)}\left[X_0, X_v\right]$$

where $P^m_{L(C)}\left[X_0, X_u\right]$ is the m-step transition probability that the system moves from the state $\left[X_0\right]^2_{L(C)}$ to the state $\left[X_v\right]^2_{L(C)}$.

3. If the components are independent and identically distributed (i.i.d.) then, the probability transient matrix is expressed as :

$$P_{L(C)} = \begin{array}{cc} \text{classes} & \begin{array}{cccccccc} [\varnothing]^2_{L(C)} & [X_1]^2_{L(C)} & [X_2]^2_{L(C)} & & [X_l]^2_{L(C)} & & [X_s]^2_{L(C)} & \Psi^2_{L(C)} \end{array} \\ \begin{array}{c} [\varnothing]^2_{L(C)} \\ [X_1]^2_{L(C)} \\ [X_2]^2_{L(C)} \\ \vdots \\ [X_l]^2_{L(C)} \\ \vdots \\ [X_s]^2_{L(C)} \\ \Psi^2_{L(C)} \end{array} & \left(\begin{array}{ccccccc|c} p^n & d^{X_1}_{X_0} R(X_1) & d^{X_2}_{X_0} R(X_2) & \cdots & d^{X_l}_{X_0} R(X_l) & \cdots & d^{X_s}_{X_0} R(X_s) & C_{X_0} \\ p^n & d^{X_1}_{X_1} R(X_1) & d^{X_2}_{X_1} R(X_2) & \cdots & d^{X_l}_{X_1} R(X_l) & \cdots & d^{X_s}_{X_1} R(X_s) & C_{X_1} \\ p^n & d^{X_1}_{X_2} R(X_1) & d^{X_2}_{X_2} R(X_2) & \cdots & d^{X_l}_{X_2} R(X_l) & \cdots & d^{X_s}_{X_2} R(X_s) & C_{X_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p^n & d^{X_1}_{X_l} R(X_1) & d^{X_2}_{X_l} R(X_2) & \cdots & d^{X_l}_{X_l} R(X_l) & \cdots & d^{X_s}_{X_l} R(X_s) & C_{X_l} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p^n & d^{X_1}_{X_s} R(X_1) & d^{X_2}_{X_s} R(X_2) & & d^{X_l}_{X_s} R(X_l) & \cdots & d^{X_s}_{X_s} R(X_s) & C_{X_s} \\ \hline 0 & 0 & 0 & & 0 & \cdots & 0 & 1 \end{array} \right)_{(s+2)\times(s+2)} \end{array}$$

Figure 3.1: The failure states of the 2-within- consecutive (2,2)-out-of-(m,n):F systems

Definition 3.1: Imbedding the 2-within-consecutive (2,2)-out-of-(m,n): F linear (rectangle) and circular (cylindrical) system in a Markov chain.

A. For any layer (circle) in the system, and without loss of generality, we can rearrange the finite failed component states $P\left(\mathrm{I}_n^1\right)$ of as $P\left(\mathrm{I}_n^1\right)=\left\{[\varnothing]_{L(C)}^2=[X_0]_{L(C)}^2,[X_1]_{L(C)}^2,[X_2]_{L(C)}^2,\ldots,[X_s]_{L(C)}^2,[X_{s+1}]_{L(C)}^2=\Psi_{L(C)}^2\right\}$, where $[X_u]_{L(C)}^2\cap[X_v]_{L(C)}^2=\varnothing:\forall u\neq v$.

B. Consider the subsystem with i layers (circles) in the functioning state. Add a new layer (circle) to the subsystem and keep it in the functioning state, then the failed components in the (i+1) th layer (circle) depend only on the failed components on the ith layer (circle), they must be away from those in the ith layer (circle) at least 2 steps.

Now, if $S_i\in P\left(\mathrm{I}_n^1\right):i=1,2,\ldots,m$ is a random variable represents any class $[X_u]_{L(C)}^2\in P\left(\mathrm{I}_n^1\right)$ $:u=0,1,\ldots,s+1$ of the ith layer (circle), then the random variable S_{i+1} depends only on S_i but not on $S_{i-1},S_{i-2},\ldots,S_1$, hence the sequence $\{S_i\},i=1,2,\ldots,m$ forms a Markov chain

- The variables $S_i:i=1,2,\ldots,m$ are defined on $P\left(\mathrm{I}_n^1\right)$ such that, $S_i=[X_u]_{L(C)}^2:u=0,1,\ldots,s$ if and only if the ith layer (circle) in the subsystem with i layers (circles) has any functioning subset $Z\in[X_u]_{L(C)}^2$.
- $S_i=[X_{s+1}]_{L(C)}^2=\Psi_{L(C)}^2$ if and only if the subsystem consisting of i layers (circles) is failed.

Theorem 3.1: Consider the 2-within (2,2)-out-of-(i,n): F linear (rectangle) and circular (cylindrical) subsystem, the state of the subsystem is determined by the state of the ith layer (circle), $S_i\in P\left(\mathrm{I}_n^1\right):i=1,2,\ldots,m$, the variable S_i

Note that $A^{i+1}_{L(C)}\left(\varnothing,[Y]^2\right)=[Y]^2_{L(C)}$ and $A^{i+1}_{L(C)}\left(X,[Y]^2\right)=\Theta^2_{L(C)}(X)\cap[Y]^2_{L(C)}$.

Lemma 3.1: $f^{\beta}_{L(C)}(X)=W\Rightarrow f^{\beta}_{L(C)}\left(A^{i+1}_{L(C)}\left(X,[Y]^2\right)\right)=A^{i+1}_{L(C)}\left(W,[Y]^2\right)$

Proof: If $Z\in f^{\beta}_{L(C)}\left(A^{i+1}_{L(C)}\left(X,[Y]^2\right)\right)\Leftrightarrow \exists H\in A^{i+1}_{L(C)}\left(X,[Y]^2\right)$ such that $Z=f^{\beta}_{L(C)}(H)\Leftrightarrow$

Circular system

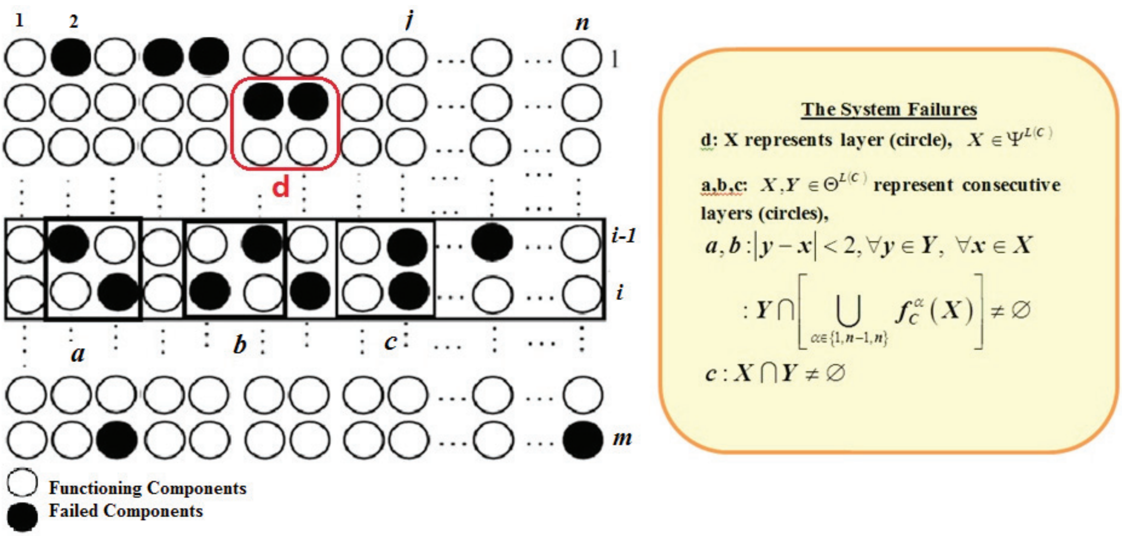
$$\Leftrightarrow Z=f^{\beta}_{C}(H)\Leftrightarrow \varnothing=H\cap\left[\bigcup_{\alpha\in\{1,n-1,n\}} f^{\alpha}_{C}(X)\right]\Leftrightarrow$$

$$\varnothing=f^{\beta}_{L(C)}(H)\cap\left[\bigcup_{\alpha\in\{1,n-1,n\}} f^{\alpha}_{C}\left(f^{\beta}_{C}(X)\right)\right]=\underbrace{f^{\beta}_{L(C)}(H)}_{Z}\cap\left[\bigcup_{\alpha\in\{1,n-1,n\}} f^{\alpha}_{C}(W)\right]\Leftrightarrow Z\in A^{i+1}_{C}\left(W,[Y]^2\right)$$

Linear system

$$\Leftrightarrow Z=f^{\beta}_{L}(H)\Leftrightarrow |h-x|\geq 2:\forall h\in H,\forall x\in X\Leftrightarrow \text{since } f \text{ is bijection}$$

$$\left|f^{\beta}_{L}(h)-f^{\beta}_{L}(x)\right|\geq 2:\forall h\in H,\forall x\in X\Leftrightarrow |z-w|\geq 2:\forall z\in Z,\forall w\in W\Leftrightarrow Z\in A^{i+1}_{L}\left(W,[Y]^2\right)$$



have a square that includes at least two connected failed components) see figure 3.1. Now, $P\left(\mathsf{I}_n^1\right)=\Theta_{L(C)}^2 \cup \Psi_{L(C)}^2$ is a union of mutual disjoint classes, where,

$$\Theta_{L(C)}^2=\left\{[\varnothing]_{L(C)}^2=\left[X_0\right]_{L(C)}^2,\left[X_1\right]_{L(C)}^2,\left[X_2\right]_{L(C)}^2,\ldots,\left[X_s\right]_{L(C)}^2\right\}=\bigcup_{j=0}^{s}\left[X_j\right]_{L(C)}^2 \Rightarrow$$

$$P\left(\mathsf{I}_n^1\right)=\Theta_{L(C)}^2 \cup \Psi_{L(C)}^2=\left(\bigcup_{j=0}^{s}\left[X_j\right]_{L(C)}^2\right) \cup \Psi_{L(C)}^2$$

.Say $\left[X_{s+1}\right]_{L(C)}^2=\Psi_{L(C)}^2 \Rightarrow P\left(\mathsf{I}_n^1\right)=\bigcup_{j=0}^{s+1}\left[X_j\right]_{L(C)}^2$ where $\left[X_{s+1}\right]_{L(C)}^2$ is the only failed class.

Consider $X \in \Theta_{L(C)}^2$ represents the ith layer (circle), define $\Theta_{L(C)}^2(X)$ to be the set of all functioning subset $Z \in \Theta_{L(C)}^2$ in the (i+1) th layer (circle) that guarantees that Z must not cause the system fail, i.e. any failed components in the layer (circles) the ith and the i+1th are away from each other at least 2 steps, otherwise the whole system fails (see figure 3.1.) i.e. $\Theta_C^2(X)=\left\{Z \in \Theta_C^2 : \mathrm{Z} \cap\left[\bigcup_{\alpha \in\{1, n-1, n\}} f_C^\alpha(X)\right]=\varnothing\right\}$ for the circular system, and $\Theta_L^2(X)=\left\{Z \in \Theta_L^2 :|z-x| \geq 2\right\}$ for the linear system. (Note that $\Theta_{L(C)}^2(\varnothing)=\Theta_{L(C)}^2$).

Also, in the same context, consider $X \in \Theta_{L(C)}^2$ represents the ith layer (circle), define $A_{L(C)}^{i+1}\left(X,[Y]^2\right)$ to be the set of all functioning subset $Z \in[Y]_{L(C)}^2$ represents the (i+1) th layer (circle) that guarantee that Z must not cause the system fail, i.e.

$$A_C^{i+1}\left(X,[Y]^2\right)=\left\{Z \in[Y]_C^2 : \mathrm{Z} \cap\left[\bigcup_{\alpha \in\{1, n-1, n\}} f_C^\alpha(X)\right]=\varnothing\right\}$$

$$A_L^{i+1}\left(X,[Y]^2\right)=\left\{Z \in[Y]_L^2 :|z-x| \geq 2\right\}$$

X is a functioning subset, $\mathrm{I}_{r+k-1}^{r} \not\subset X \Rightarrow f_{L}^{\alpha}\left(\mathrm{I}_{r+k-1}^{r}\right) \not\subset f_{L}^{\alpha}\left(X\right)=Y$. If α is even $f_{L}^{\alpha}\left(\mathrm{I}_{r+k-1}^{r}\right)=\mathrm{I}_{r+k-1}^{r} \not\subset f_{L}^{\alpha}\left(X\right)=Y$ and if it's odd $f_{L}^{\alpha}\left(\mathrm{I}_{r+k-1}^{r}\right)=\mathrm{I}_{n+1-r}^{n-(r+k)+2} \not\subset f_{L}^{\alpha}\left(X\right)=Y$,which implies that Y is a functioning subset.

For the circular system, if $Y \in [X]_{C}^{k} \Rightarrow \exists \alpha \in \mathbf{Z}$ such that $f_{L}^{\alpha}\left(X\right)=Y$, since X is a functioning subset,

$$\bigcup_{\beta=0}^{k-1} f^{\beta}\left(r\right) \not\subset X \Rightarrow f_{C}^{\alpha}\left(\bigcup_{\beta=0}^{k-1} f^{\beta}\left(r\right)\right)=\bigcup_{\beta=0}^{k-1} f^{\beta}\left(f^{\alpha}\left(r\right)\right)=\bigcup_{\beta=0}^{k-1} f^{\beta}\left(\left(r \bmod n\right)+1\right) \not\subset f_{C}^{\alpha}\left(X\right)=Y$$

, which implies that, Y is a functioning subset (replace $\not\subset$ by $\subseteq$ for the proof of failed subset).

Lemma 2.2: In the consecutive k-out-of-n: F linear (circular) system, if the failed components are represented by $Z \in [X]_{L(C)}^{k}$, such that $p_{j}=p: j=1,2,\ldots,n$, then $R\left(Z\right)=R\left(X\right)$.

Proof: If $Z \in [X]_{L(C)}^{k} \Leftrightarrow \exists \alpha \in \mathbf{Z}$ such that $Z=f_{L(C)}^{\alpha}\left(X\right)$, since $f_{L(C)}^{\alpha}$ is a bijection function, then $d_{Z}=d_{X}$. Also $p_{j}=p$, it implies that,

$$R\left(Z\right)=p_{Z}q_{Z}=\prod_{j\notin Z} p_{i}\prod_{j\in Z} q_{j}=p^{n-d_{Z}}q^{d_{Z}}=p^{n-d_{X}}q^{d_{X}}=\prod_{j\notin X} p_{i}\prod_{j\in X} q_{j}=p_{X}q_{X}=R\left(X\right)$$

.Imbedding the 2-within-consecutive (2,2)-out-of-(m,n): F linear (rectangle) and circular (cylindrical) systems in a Markov chain.

Consider the 2-within-consecutive (2,2)-out-of-(m,n):F linear (rectangle) and circular (cylindrical) system, and I_{n}^{1} be the indices of the components at the ith layer (circle), then the failure space of the components of the ith layer (circle) is $P\left(\mathrm{I}_{n}^{1}\right)$.

If the system is in the functioning state, then any layer (circle) of the system has a functioning subset $X \in \Theta_{L(C)}^{2}$, otherwise the system fails (we

2. For the circular system, $\Theta_C = \left\{ X \in P\left(\mathrm{I}_n^1\right) : \bigcup_{\alpha=0}^{k-1} f^{\alpha}\left(r\right) \not\subset X ; r \in \mathrm{I}_n^1 \right\}$ is the functioning space and Ψ^C is the failed space.

Consider the symmetric property of the components in the consecutive k-out-of-n: F linear and circular system and define a bijection functions for the linear and circular system respectively, $f_L, f_C : \mathrm{I}_n^1 \to \mathrm{I}_n^1$ where $f_L\left(r\right) := n+1-r$, for any $r \in \mathrm{I}_n^1$ and $f_C\left(r\right) := \left(r \bmod n\right)+1$ for any $r \in \mathrm{I}_n^1$, also define the two equivalence relations for any $X, Y \in P\left(\mathrm{I}_n^1\right)$, $X \sim \left(\equiv\right) Y \Leftrightarrow \exists \alpha \in \mathbf{Z}$ such that $Y = f_{L(C)}^{\alpha}\left(X\right)$ [9-10]. According to the set theory, $P\left(\mathrm{I}_n^1\right)$ can be written as a union of a finite partition of mutually disjoint classes $[X]_{L(C)}^k = \left\{ Y \in P\left(\mathrm{I}_n^1\right) : X \sim (\equiv) Y \right\}$, consequently, $\Theta_{L(C)}^k$ also may be written as a union of a finite partition of mutually disjoint classes. If s+1 is the number of these classes, then $\Theta_{L(C)}^k = \left\{ [\varnothing]_{L(C)}^k = [X_0]_{L(C)}^k, [X_1]_{L(C)}^k, [X_2]_{L(C)}^k, \ldots, [X_s]_{L(C)}^k \right\} = \bigcup_{u=0}^{s} [X_u]_{L(C)}^k$, therefore, $R_{L(C)}^k$ the reliability of the consecutive k-out-of-n: F linear (circular) system can be written as a sum of reliability of these disjoint functioning classes.

$$R_{L(C)}^k = \sum_{[X_u]_{L(C)}^k \in \Theta_{L(C)}^k} R\left[X_u\right]_{L(C)}^k = \sum_{u=0}^{s} R\left[X_u\right]_{L(C)}^k = \sum_{u=0}^{s} \sum_{Z \in [X_u]_{L(C)}^k} R\left(Z\right)$$

where $R\left[X_u\right]_{L(C)}^k$ is the reliability of the class $\left[X_u\right]_{L(C)}^k$.

Lemma 2.1: let X is a functioning (failed) subset of the consecutive k-out-of-n: F linear (circular) system, if $Y \in [X]_{L(C)}^k$, then Y is also a functioning (failed) subset.

Proof:

For the linear system, if $Y \in [X]_L^k \Rightarrow \exists \alpha \in \mathbf{Z}$ such that $f_L^{\alpha}\left(X\right) = Y$, since reliability, Chang & Huang [4] evaluated the reliability of the system using 2 dimensional scan statistic and finite Markov chain approach.

In this paper, a new algorithm is obtained to imbed the 2-within-consecutive (2,2)-out-of-(m,n): F linear (rectangle) and circular (cylindrical) systems in a Markov chain; which can be expressed the reliability of the system in a simple form in terms of the transition probability matrix.

The following assumptions are assumed to be satisfied by the 2-within-consecutive (2,2)-out-of-(m,n): F rectangle (cylindrical) system:

1. The state of the component and the system is either " functioning" or "failed"

2. All the components are mutually statistically independent.

3. The consecutive k-out of n: F linear and circular system

In this section, the Index Structure Function [9] is used to present the consecutive k-out-of-n: F system. Let I_n^1 denotes the indices of the components of the consecutive k-out-of-n: F linear and circular system, and $X \subseteq I_n^1$ denotes the indices of the failed components. If the system is in the functioning state; we call the set X "functioning subset"; otherwise it is "failed subset", (e.g. in the consecutive 2-out-of 6: linear and circular system, the subset $X = \{13\} \subset I_6^1$ or for simply 13, indicates that the 1st and the 3rd components are failed. In spite of these failed components, the system stills in the functioning state, so 13 is a functioning subset. On the contrary, the subset $12 \subset I_n^1$ is a failed subset).

Now, according to probability theorem, the failure space of the components of the consecutive k-out-of-n: F linear (circular) system is $P\left(I_n^1\right)$, we can divide it into two sub collection, $\Theta^{L(C)}$ and $\Psi^{L(C)}$ the functioning and failure space respectively

1. For the linear system, $\Theta^L = \left\{X \in P\left(I_n^1\right) : I_{r+k-1}^r \not\subset X;\ \ r \in I_{n-k+1}^1\right\}$ is the functioning space, and Ψ^L is the failed space.

Introduction:

The consecutive k-out-of-n: F system is one of the most frequently studied system, due to its important applications in various engineering systems, it has a high reliability and low expense, (e.g. Telecommunication system with n relay stations, the pipeline of transmit oil system, etc.[3]). The consecutive k-out-of-n: F system consists of n components; it fails if at least k consecutive components fail. It is classified according to the connection between components into two types: linear and circular. Kontoleon's (1980) [6] was the first which studied the system under the name "r-successive-out-of-n: F system".

In 1986, Griffth [5] has introduced a generalization of the consecutive k-out-of-n: F system, "The consecutive k-within-m-out-of-n: F system", the system fails if there are m consecutive components, which include among them at least k failed components. When k=m, it becomes a consecutive k-out-of n: F system, the system has widespread applications in radar detection, quality control, acceptance sampling, safety monitoring systems, and DNA sequencing. An efficient lower and upper bounds were proposed by Sfakianikis et al. [13] and Papastavridis and Koutras [11].

In 1990, Salvia and Lasher [12] introduced a 2-dimensional consecutive system, Boehme et al. [2] present an application of the 2-dimensional consecutive system "the super vision system, and then many applications appeared like disease diagnosis on the X-ray, pattern detection.

This model was extended to linear or circular two dimensional k-within-consecutive (r,s) out-of-(m,n) : F systems, it consists of **m x n** components arranged like the elements of a **m x n** matrix (located on the intersections of m circles and n rays) and fails if and only if there are at least k failed components in a sub matrix **r x s** components. Many applications were appeared "the alarm systems" and "liquid crystal screen on a computer" [1] etc. The XGA (1024×768 = total 786432 dot) TFT display system fails if and only if more than or equal to 10 dot fail in 10×10 dot matrix, then the system become to be linear 10-within (10,10)-out-of (1024, 768): F system.

Makri and Psikallis [7] have provided an upper & lower bounds of its reliability, Akiba and Yamamoto [1] presented an algorithm for exact

Notations:

L(C)	: Linear (Circular)
I_j^i	: The set $\{i, i+1, \ldots, j\}$
$P\left(\mathrm{I}_n^1\right)$	: The power set of I_n^1
$f^{\alpha}(X)$	: $f\left(f \ldots\ldots\left(f\left(X\right)\right)\right)$ composition function α times
d_X	: The cardinality (number of elements) of the set X.
$p_i\left(q_i\right)$	: The reliability (unreliability) of the ith component
$p_{ij}\left(q_{ij}\right)$	: The reliability (unreliability) of the jth component at the ith layer (circle).
$p_Z\left(q_Z\right)$	: $p_Z = \prod_{j \notin Z} p_i , q_Z = \prod_{j \in Z} q_j$
$R(Z)$	: $= p_Z q_Z$
$p_Z^i\left(q_Z^i\right)$	: The reliability (unreliability) of the ith layer (circle), when the indices of failed components labeled by the set Z, $p_Z^i = \prod_{j \notin Z} p_{ij} , q_Z^i = \prod_{j \in Z} q_{ij}$
$P_{L(C)}$	: The transient probability Matrix of the 2-within consecutive (2,2)-out-of-(m,n): F linear (circular) system.
$R_{L(C)}(m)$	: The Reliability of the 2-within (2,2)-out-of-(m,n): F rectangle (cylindrical) system.

Abstract :

The 2-dimensional k-within-consecutive (r,s)-out-of-(m,n): F linear (rectangle) and circular (cylindrical) system fails if there is at least k failed components through the sub matrix $r \times s$ components. For example, the 2-within-consecutive (2,2)-out-of-(m,n): F linear (rectangle) and circular (cylindrical) system fails if there is at least 2 failed components through any sub matrix 2×2 components.

In this paper, a new algorithm is obtained to imbed the 2-within-consecutive (2,2)-out-of-(m,n): F rectangle (cylindrical) system in a Markov chain; this gives the possibility for computing the reliability in terms of the transition probabilities matrix of the considered Markov chain. Furthermore, the computational process of the reliability of the cylindrical system is simpler than the rectangle system since the number of states of the Markov chain in the cylindrical case is less than that in the rectangle case.

Keywords: Consecutive k-out-of-n: F system, connected (r,s)-out-of-(m,n): F system, Markov Chain, Transition probability matrix.

موثوقية النظام
"2-within Consecutive (2,2)-out-of-(m,n): F"
باستخدام سلاسل ماركوف

ملخص:

النظام التتابعي ذو البعد الثنائي (k-within-consecutive (r,s)-out-of-(m,n): F) الدائري والمستطيل يفشل اذا فشل عدد k مكون خلال مصفوفة مكونة من r X s مكون، فمثلا النظام 2-within-consecutive (2,2)-out-of-(m,n): F الدائري والمستطيل يفشل في حال فشل مكونين خلال مربع مكون من 2X2 مكون.

في هذا البحث تم إيجاد اقتران موثوقية النظام (2-within-consecutive (2,2)-out-of-(m,n): F) الدائري والمستطيل من خلال سلاسل ماركوف، هذا أعطى الإمكانية لايجاد مصفوفة الانتقال. وأكثر من ذلك فإن إيجاد اقتران موثوقية النظام الأسطواني أسهل من النظام المستطيل، وذلك لأن حالات سلاسل ماركوف بالنظام الأسطواني اقل.

كلمات مفتاحية: النظام التتابعي (k-out-of-n: F system) الخطي والدائري، سلاسل ماركوف، مصفوفة انتقال الاحتمالات

Reliability of 2-within Consecutive (2,2)-out-of-(m,n) : F Systems Using Markov Chain *

Imad I. H. Nashwan **

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** Assistant professor/ Faculty of Education/ Al Quds Open University/ Palestine.