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$$f^{(-1)}(x) = \sum_{i=-\infty}^{+\infty} a_i \frac{x^{i+1}}{(i+1)!} = \cdots + a_{-2}\frac{x^{-1}}{(-1)!} + a_{-1}\frac{x^0}{0!} + a_0\frac{x}{1!} + a_1\frac{x^2}{2!} + \cdots$$

$$= a_{-1}\frac{x^0}{0!} + a_0\frac{x}{1!} + a_1\frac{x^2}{2!} + a_2\frac{x^3}{3!} + \cdots$$

$$= a_{-1} + \int_0^x f(t)\,dt = \int f(x)\,dx$$

In general, if n is a positive integer, then

$$f^{(-n)}(x) = \sum_{i=-\infty}^{+\infty} a_i \frac{x^{i+n}}{(i+n)!} = \cdots + a_{-n-1}\frac{x^{-1}}{(-1)!} + a_{-n}\frac{x^0}{0!} + a_{-n+1}\frac{x}{1!} + a_{-n+2}\frac{x^2}{2!} + \cdots$$

$$= a_{-n}\frac{x^0}{0!} + a_{-n+1}\frac{x}{1!} + a_{-n+2}\frac{x^2}{2!} + \cdots + a_0\frac{x^n}{n!} + \cdots$$

$$= a_{-n} + a_{-n+1}\frac{x}{1!} + a_{-n+2}\frac{x^2}{2!} + \cdots + a_{-1}\frac{x^{n-1}}{(n-1)!} + \int_0^x\int_0^x\cdots\int_0^x f(t)\,dt,$$

n-times integrations.

Hence, $f^{(-n)}(x) = \int\int\cdots\int f(x)\,dx$, n-times integrations.

Now, if we take $f(x) = e^x$, then for any positive integer n, we have

$f^{(n)}(x) = e^x$ and $f^{(-n)}(x) = \int\int\cdots\int e^x\,dx$

Therefore, can we prove that $f^{(\alpha)}(x) = e^x$ for any $\alpha \in R$ by using this method? In [2], and [6] , they proved that it is true via the method described in Equation (1.1). Also, we feel the answer is true by our method and we conjecture the following:

Conjecture:

For any $\alpha \in R$, $\dfrac{d^{(\alpha)}}{dx^{(\alpha)}}\left(e^x\right) = e^x$.

$$= \sum_{i=-\infty}^{+\infty} a_i \frac{x^{i-\alpha}}{(i-\alpha)!} + \sum_{i=-\infty}^{+\infty} b_i \frac{x^{i-\alpha}}{(i-\alpha)!}$$

$$= f^{(\alpha)}(x) + g^{(\alpha)}(x)$$

(ii) Since $f(x) = \sum_{i=-\infty}^{+\infty} a_i \frac{x^i}{i!}$, applying **Definition 1** to the function $f(x)$, we get

$$f^{(\alpha)}(x) = \sum_{i=-\infty}^{+\infty} a_i \frac{x^{i-\alpha}}{(i-\alpha)!}$$

Applying **Definition 1** to the function $f^{(\alpha)}(x)$, we get

$$\frac{d^{\beta}}{dx^{\beta}}\left(f^{(\alpha)}(x)\right) = \sum_{i=-\infty}^{+\infty} a_i \frac{x^{i-\alpha-\beta}}{((i-\alpha)-\beta)!}$$

But

$$\frac{d^{\alpha+\beta}}{dx^{\alpha+\beta}}\left(f(x)\right) = \sum_{i=-\infty}^{+\infty} a_i \frac{x^{i-(\alpha+\beta)}}{((i-(\alpha+\beta))!} = \sum_{i=-\infty}^{+\infty} a_i \frac{x^{i-\alpha-\beta}}{(i-\alpha-\beta)!}$$

Therefore,

$$\frac{d^{\alpha}}{dx^{\alpha}}\left(\frac{d^{\beta}(f(x))}{dx^{\beta}}\right) = \frac{d^{\alpha+\beta}(f(x))}{dx^{\alpha+\beta}}.$$

(iii) Since $f(x) = \sum_{i=-\infty}^{+\infty} a_i \frac{x^i}{i!}$, then $cf(x) = \sum_{i=-\infty}^{+\infty} ca_i \frac{x^i}{i!}$. Applying **Definition 1** to the function $cf(x)$, we get

$$(cf)^{(\alpha)}(x) = \sum_{i=-\infty}^{+\infty} ca_i \frac{x^{i-\alpha}}{(i-\alpha)!} = c\sum_{i=-\infty}^{+\infty} a_i \frac{x^{i-\alpha}}{(i-\alpha)!} = cf^{(\alpha)}(x) \qquad \blacksquare$$

Note that for any positive integer *n*, this definition agrees with the traditional definition of the derivative as the following examples show:

(1) $\frac{d^n}{dx^n}(c) = \frac{c}{\Gamma(1-n)} x^{-n} = \frac{c}{(-n)!} x^{-n} = 0$

(2) $\frac{d^3}{dx^3}(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-3+1)} x^{n-3} = \frac{n!}{(n-2)!} x^{n-3} = n(n-1)x^{n-3}$.

Also, if $\alpha = 0$ we have

$$f^{(0)}(x) = \sum_{i=-\infty}^{+\infty} a_i \frac{x^{i-0}}{(i-0)!} = \sum_{i=-\infty}^{+\infty} a_i \frac{x^i}{(i)!} = f(x).$$

Now the question is the case *n* is a negative derivative integer? let $n = -1$, then

$$f^{(\alpha)}(x)=\sum_{i=-\infty}^{+\infty} a_i \frac{x^{i-\alpha}}{(i-\alpha)!}=a_n \frac{x^{n-\alpha}}{(n-\alpha)!}=n!\frac{x^{n-\alpha}}{(n-\alpha)!}=\frac{\Gamma(1+n)}{\Gamma(n-\alpha+1)!}x^{n-\alpha}. \qquad \blacksquare$$

Remark: In [2], it is proved that, by using the definition as in Equation (1.1),

$$\frac{d^\alpha}{dx^\alpha}(c)=\frac{c}{\Gamma(1-\alpha)}x^{-\alpha}$$

$$\frac{d^\alpha}{dx^\alpha}(x^n)=\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{n-\alpha}$$

Therefore, our definition to the fractional derivatives of the constant function and the function x^n give the same results.

4. SOME PROPERTIES

We are now in a position to discuss some properties of fractional derivatives. We will show that fractional derivatives have several properties that one would expect, such as the fractional derivative operator is linear and repeated fractional differentiation is accumulative.

Theorem 2:

If $f(x)=\sum_{i=-\infty}^{+\infty} a_i \frac{x^i}{i!}$ and $g(x)=\sum_{i=-\infty}^{+\infty} b_i \frac{x^i}{i!}$ for each *x*. Then for any $\alpha,\beta,c\in R$,

(i) $\frac{d^\alpha}{dx^\alpha}\big(f(x)+g(x)\big)=f^{(\alpha)}(x)+g^{(\alpha)}(x)$.

(ii) $\frac{d^\alpha}{dx^\alpha}\left(\frac{d^\beta(f(x))}{dx^\beta}\right)=\frac{d^{\alpha+\beta}(f(x))}{dx^{\alpha+\beta}}$.

(iii) $\frac{d^\alpha}{dx^\alpha}\big(cf(x)\big)=cf^{(\alpha)}(x)$.

Proof:

(i) Since $f(x)=\sum_{i=-\infty}^{+\infty} a_i \frac{x^i}{i!}$ and $g(x)=\sum_{i=-\infty}^{+\infty} b_i \frac{x^i}{i!}$. Then

$$f(x)+g(x)=\sum_{i=-\infty}^{+\infty}\left(a_i+b_i\right)\frac{x^i}{i!}$$

Applying **Definition 1** to the function $f(x)+g(x)$, we get

$$(f(x)+g(x))^{(\alpha)}(x)=\sum_{i=-\infty}^{+\infty}\left(a_i+b_i\right)\frac{x^{i-\alpha}}{(i-\alpha)!}$$

(1.3) $$f^{(\alpha)}(x) = \sum_{i=-\infty}^{+\infty} a_i \frac{x^{i-\alpha}}{(i-\alpha)!}$$ ■

Notes:

(i) For each $x > 0$, $x! = \Gamma(x+1)$.

(ii) $(-\alpha)! = \dfrac{\Gamma(-\alpha+m)}{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+m-1)}$, $m-1 < \alpha < m$, m is a positive integer.

(iii) For each $x \neq 0$, $\dfrac{x^{i-\alpha}}{(i-\alpha)!} \neq 0$ if α is non-integer number.

The interpretation of the coefficients $a_{-i}, i \in Z^+$ is the following. The coefficient a_{-1} is equal to $g(0)$, where $g'(x) = f(x)$, i.e. it is the integral constant of $\int f\, dx$. Similarly, a_{-2} is equal to the integral constant of $\int\left(\int f\, dx\right)dx$, and so on.

Thus the calculation of the fractional derivative of f requires knowledge of all coefficients of integration, but not only the mapping $f: R \to R$. Note that if f is given by the right side of (1.2), then its derivatives of any order is defined similarly. It happens very often that the left side of (1.3) diverges, and then we apply the method of summation of the divergent series presented previously.

3. APPLICATIONS

Using this new approach we can prove the following theorem that verifies some formulas for fractional derivatives of order $\alpha \in R$ of several functions.

Theorem 1:

(i) If $f(x) = c$, where c is a constant, then $f^{(\alpha)}(x) = c\dfrac{x^{-\alpha}}{(-\alpha)!} = \dfrac{c}{\Gamma(1-\alpha)} x^{-\alpha}$.

(ii) If $f(x) = x^n$, $n \in Z$, then $f^{(\alpha)}(x) = \dfrac{n!x^{n-\alpha}}{(n-\alpha)!} = \dfrac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}$.

Proof:

(i) Write $f(x) = \sum_{i=-\infty}^{+\infty} a_i \dfrac{x^i}{i!}$, where $a_i = 0, \forall i \neq 0$ and $a_0 = c$. Applying **Definition 1** to the function f, we get

$$f^{(\alpha)}(x) = \sum_{i=-\infty}^{+\infty} a_i \frac{x^{i-\alpha}}{(i-\alpha)!} = a_0 \frac{x^{0-\alpha}}{(0-\alpha)!} = c\frac{x^{-\alpha}}{(-\alpha)!} = c\frac{x^{-\alpha}}{\Gamma(1-\alpha)}.$$

(ii) Write $f(x) = \sum_{i=-\infty}^{+\infty} a_i \dfrac{x^i}{i!}$, where $a_i = 0, \forall i \neq n$ and $a_n = n!$. Applying **Definition 1** to the function f, we get

1. INTRODUCTION

A fractional derivative $\frac{d^{\alpha} f(t)}{dt^{\alpha}}$ is an extension of the familiar *n*th derivative $\frac{d^{n} f(t)}{dt^{n}}$ of the function $f(t)$. The literature contains many examples of the use of fractional derivatives, see [1],[3]. The most common definition for the fractional derivative of order $\alpha \in R$ of a function *f* is the "Riemann-Liouville integral", see [2],[3],[4],[7]:

(1.1)
$$\frac{d^{\alpha} f(x)}{dx^{\alpha}} = f^{(\alpha)}(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{f(t)}{(x-t)^{\alpha+1}} dt$$

where $\Gamma(n)$ is the Euler's Gamma Function.

For example, using equation (1.1), the (1/2)th derivatives of the functions $f(x) = x$ and $g(x) = \sqrt{x}$ can be evaluated as

$$\frac{d^{1/2}}{dx^{1/2}} x = \frac{2\sqrt{x}}{\sqrt{\pi}} \quad \text{and} \quad \frac{d^{1/2}}{dx^{1/2}} \sqrt{x} = \frac{\sqrt{\pi}}{2}$$

In this paper, we will consider summation of "divergent" series for calculating of fractional derivatives of order $\alpha \in R$ of several functions. We give some examples and prove some properties for the fractional derivatives.

2. DISCUSSION OF THE METHOD

Let $\sum_{i=0}^{+\infty} b_i$ be a given series. Consider the formal power series $\sum_{i=0}^{+\infty} b_i x^i$ and look for a differential equation which it satisfies, even if the radius of convergence of the power series is 0. If f is the solution of the corresponding differential equation, then we take $f(x) = \sum_{i=0}^{+\infty} b_i x^i$ for each *x*. Set $f(x) := \sum_{i=0}^{+\infty} a_i \frac{x^i}{i!}$ and define $\frac{x^i}{i!} = 0$ for each *x* and for $i = -1, -2, \cdots$, then we can write f in the form

(1.2)
$$f(x) = \sum_{i=-\infty}^{+\infty} a_i \frac{x^i}{i!}$$

Definition 1:

If $f(x) = \sum_{i=-\infty}^{+\infty} a_i \frac{x^i}{i!}$ for each *x*. Then, for any $\alpha \in R$, the fractional derivative of order $\alpha \in R$ of a function *f* is defined to be

ملخص

موضوع هذا البحث هو مشتقة اقتران ما من الدرجة α حيث أن αعدد حقيقي ليس بالضرورة أن يكون عددا صحيحا موجبا (المشتقة الكسرية) . لقد قمنا في هذا البحث بتقديم تعريف جديد للمشتقة الكسرية لاقتران ما بالاعتماد على متسلسلة القوة للاقتران .

وفي هذا السياق واعتمادا على تعريفنا الجديد قمنا بحساب المشتقة الكسرية للاقترانات الثابتة والاقترانات كثيرة الحدود حيث كانت النتائج مطابقة للنتائج الناتجةعن تطبيق ما هو معروف الى الان من تعريفات للمشتقة الكسرية . اضافة الى ذلك قمنا باثبات بعض خصائص المشتقة الكسرية .

وأخيرا فاننا نعتقد بان تكون المشتقة الكسرية للاقتران الاسي e^x هي الاقتران الاسي e^x نفسه وهذا يساعدنا في الحصول على المشتقة الكسرية للاقترانات المثلثية والاقترانات الزائدية .

ان اهمية هذا البحث تكمن في ايجاد الية سهلة التطبيق لايجاد المشتقات الكسرية للاقترانات وحيث ان كل اقتران قابل للاشتقاق يمكن كتابته على طريقة متسلسلة القوى بغض النظر عن التقارب او عدمه فاننا نرى ان هذه الطريقة يمكن أن تعطينا نتائج طيبة في هذا المجال والذي بدوره سوف ينعكس على التطبيقات العملية الكثيرة .

Abstract

A derivative of a function of order α , for any real number α (called a fractional derivative) is the subject of this paper. In this paper a new definition of the fractional derivative of order $\alpha \in R$ of a function is given. This new definition will depend on the formal power series summation. We used this new definition to find the fractional derivative of the constant functions and the polynomials. The result was the same result by using the known definitions of fractional derivatives until now. Also, we proved properties of the fractional derivative. Finally, we conjecture that the fractional derivative of order $\alpha \in R$ of the exponential function e^x is the exponential function e^x and this will help us in finding the fractional derivatives of the trigonometric functions and hyperbolic functions.

The purpose of this research is to find an easy way to derive the fractional derivatives for the functions, since every differentiable function can be represented by power series expansion, even if the radius of convergence of the power series is 0, we believe that this method will give nice results which can be used in the application.

A NEW APPROCH TO FRACTIONAL DERIVATIVES

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